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GENERALIZED QUANTIFIERS AND COMPACT LOGIC

BY

SAHARON SHELAH

ABSTRACT. We solve a problem of Friedman by showing the existence of a logic stronger than first-order logic even for countable models, but still satisfying the general compactness theorem, assuming e.g. the existence of a weakly compact cardinal. We also discuss several kinds of generalized quantifiers.

Introduction. We assume the reader is acquainted with Lindström's articles **[Li 1]** and **[Li 2]** where he defined "abstract logic" and showed in this framework simple characterizations of first-order logic. For example, it is the only logic satisfying the compactness theorem and the downward Löwenheim-Skolem theorem. Later this was rediscovered by Friedman **[Fr 1]**; and Barwise **[Ba 1]** dealt with characterization of infinitary languages.

Keisler asked the following question:

(1) Is there a compact logic (i.e., a logic satisfying the compactness theorem) stronger than first-order logic? It should be mentioned that it is known for many $L(Q_{\aleph_\alpha})$ that they satisfy the λ-compactness theorem for $\lambda < \aleph_\alpha$ (for $\alpha > 0$). ($Q_{\aleph_\alpha}(x) \Longleftrightarrow$ there are $\geqslant \aleph_\alpha$ x's; the λ-compactness theorem says that if T is a theory in $L(Q_{\aleph_\alpha})$, $|T| \leqslant \lambda$, and for all finite $t \subseteq T$ there is a model, then T has a model.) For example, this is the case for $\alpha = 1$. See Fuhrken **[Fu 1]**, Keisler **[Ke 2]** and see **[CK]** for general information.

At the Cambridge summer conference of 1971 Friedman asked:

(2) Is there a logic satisfying the compactness theorem, or even the $\aleph_0$-compactness theorem, which is stronger than first-order logic even for *countable* models, i.e., is there a sentence ψ in the logic such that there is no first order sentence φ such that for all countable models M, $M \models \psi \Longleftrightarrow M \models \varphi$?

Notice that the power quantifiers $Q_{\aleph_\alpha}$ do not satisfy the second part of (2). The quantifier saying "$\varphi(x, y)$ is an ordering with cofinality $\aleph_1$" solves (1) (but obviously not (2)) as proved, in fact in **[Sh 2, §4.4]** and noticed by me in Cambridge.

The main result of this paper is the presentation in §1 of an example solving both (1) and (2) positively (assuming the existence of a weakly compact cardinal); thus, compactness alone does not characterize first-order logic. In §2 we mention

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all kinds of problems about generalized second-order quantifiers, and prove some results.

After the solution Friedman asked:

(3) Is there a compact logic, stronger than first-order logic even for finite models?

Notation. $\lambda, \mu, \kappa, \chi$ designate cardinals; $i, j, k, l, \alpha, \beta, \gamma, \delta, \xi$ designate ordinals; and m, n are natural numbers. The power of A is $|A|$. Models are M, N, and the universe of M is $|M|$. a, b, c are elements; $\bar{a}, \bar{b}, \bar{c}$ finite sequences of elements; $l(\bar{a})$ is the length of the sequence $\bar{a}$. x, y, z, v will be variables, and $\bar{x}, \bar{y}, \bar{z}, \bar{v}$ sequences of variables.

1. A compact logic different from first-order logic.

The following theorem is proven under the assumption of the existence of a weakly compact cardinal (see Silver [Si 1]).

THEOREM 1.1. *(There is a weakly compact cardinal κ.) There is a compact logic L^*, which is stronger than first-order logic even for countable models.*

DEFINITION 1.1. $\mathrm{cf}(A, <)$, the cofinality of the ordering $<$ on the set A, is the first cardinal λ such that there exists $B \subseteq A$, $|B| = \lambda$, B is unbounded from above in A. $\mathrm{cf}^*(A, <)$ is $\mathrm{cf}(A, >)$, $>$ the reverse order. When $<$ is understood we just write $\mathrm{cf}(A)$ or $\mathrm{cf}^*(A)$. It is easy to see that the cofinality is a regular cardinal (or 0 or 1).

DEFINITION 1.2. (A_1, A_2) is a Dedekind cut of the ordered set $(A, <)$ (or just cut for short) if $A_1 \cup A_2 = A; b_1 \in A_1 \wedge b_2 \in A_2 \rightarrow b_1 < b_2; b < b_1 \in A_1 \rightarrow b \in A_1$.

DEFINITION 1.3. Let C be a class of regular cardinals. We shall define two generalized quantifiers $(Q_C^{\mathrm{cf}} x, y)$ and $(Q_C^{\mathrm{dc}} x, y)$:

(A) $M \models (Q_C^{\mathrm{cf}} x, y)\varphi(x, y; \bar{a}) \Longleftrightarrow$ the relation $x < y \equiv_{\mathrm{def}} \varphi(x, y; \bar{a})$ linearly orders $A = \{b \in M: M \models (\exists x)\varphi(x, b; \bar{a})\}$ and $\mathrm{cf}(A, <) \in C$.

(B) $M \models (Q_C^{\mathrm{dc}} x, y)\varphi(x, y; \bar{a}) \Longleftrightarrow$ the relation $x < y \equiv_{\mathrm{def}} \varphi(x, y; \bar{a})$ linearly orders $A = \{b \in M: M \models (\exists x)\varphi(x, b; \bar{a})\}$ and there is a Dedekind cut (A_1, A_2) of $(A, <)$ such that $\mathrm{cf}(A_1, <), \mathrm{cf}^*(A_2, <) \in C$. Clearly the syntax of $L(Q_C^{\mathrm{cf}}, Q_C^{\mathrm{dc}})$, the logic obtained by adding the two generalized quantifiers to first-order logic, is not dependent on C.

DEFINITION 1.4. $L^* = L(Q^{\mathrm{cf}}_{\{\aleph_0, \kappa\}}, Q^{\mathrm{dc}}_{\{\aleph_0, \kappa\}})$ where κ is the first weakly compact cardinal. In the following we shall omit writing $\{\aleph_0, \kappa\}$.

LEMMA 1.2. *L^* is stronger than L for countable models.*

PROOF. We must find a sentence $\psi \in L^*$ for which there is no $\psi' \in L$ such that for every countable model $M, M \models \psi \Longleftrightarrow M \models \psi'$.

Let $\psi = [<$ is a linear order$] \wedge$ [every element has an immediate follower and an immediate predecessor] $\wedge \neg (Q^{\mathrm{dc}}x, y)(x < y)$.

Clearly a countable order satisfies ψ iff it is isomorphic to the order of the integers. So clearly there is no sentence of L equivalent to ψ for countable models.

THEOREM 1.3. *L^* is compact.*

REMARK. If we just wanted to prove λ-compactness for $\lambda < \kappa$, the proof would be somewhat easier.

In order to take care of the possibility that $|L| \geqslant \kappa$, we encode all the m-place relations by one relation with parameters and then we use saturativity. A similar trick was used by Chang [Ch 2] who attributes it to Vaught who attributes it [Va 1] to Chang.

We also use the technique of indiscernibles from Ehrenfeucht-Mostowski [EM]. Helling [He 1] used indiscernibles with weakly compact cardinals.

PROOF OF THEOREM 1.3. Let T be a theory in L^* such that every finite subtheory $t \subseteq T$ has a model. We must show that T has a model. Without loss of generality we may make the following assumptions.

Assumption 1. There is a singular cardinal $\lambda_0 > |T| + \kappa$ such that every (finite) $t \subseteq T$ has a model of power λ_0. (There is clearly a singular $\lambda_0 > \kappa + |T|$ such that every $t \subseteq T$ has a model of power $< \lambda_0$. Now let P be a new one-place predicate symbol, and replace every sentence of T by its relativization to P (i.e. replace $(Q^{\mathrm{cf}}x, y)\varphi(x, y, \bar{z})$ by $(Q^{\mathrm{cf}}x, y)(P(x) \wedge P(y) \wedge \varphi(x, y, \bar{z}))$ and replace $(Q^{\mathrm{dc}}x, y)\varphi(x, y, \bar{z})$ by $(Q^{\mathrm{dc}}x, y)(P(x) \wedge P(y) \wedge \varphi(x, y, \bar{z})))$. Let T' be the resulting theory. Clearly every $t \subseteq T'$ has a model of power λ_0, and T' has a model iff T has a model. Also $|T'| = |T|$.

Assumption 2. Every $t \subseteq T$ has a model M_t (of power λ_0) whose universe set is $\lambda_0 = \{\alpha: \alpha < \lambda_0\}$, $<$ (the order on the ordinals) is a relation of M_t, $RC^{M_t} = \{\mu: \mu < \lambda_0$ is a regular cardinal$\}$, ω and κ are individual constants, and there is a pairing function.

Assumption 3. There is $L_a \subseteq L$, L_a countable, and the only symbols in $L - L_a$ are individual constants, and ω, κ are in L_a. We can assume that L has no function symbols.

Let $\{R_i^n: i < \alpha_n, n < \omega\}$ be a list of all the predicate symbols in L, R_i^n being n-place. Define languages L'_0, L'_1 as follows: $L'_1 = \{\omega, \kappa, <\} \cup \{R^n: n < \omega, R^n$ is an $(n + 1)$-place predicate symbol$\}$, $L'_0 = L'_1 \cup \{c_i^n: i < \alpha_n, n < \omega, c_i^n$ individual constant symbol$\}$. If $\psi \in T$ define ψ_0 by replacing every occurrence of $R_i^n(x_1, \cdots, x_n)$ in ψ by $R^n(x_1, \cdots, x_n, c_i^n)$. Let $T_0 = \{\psi_0: \psi \in T\}$, T_0 is a theory in L'^*_0 and may be taken in place of T.

Claim 1.4. For every language L_b containing $<$ there is a language L_c and a theory $T_c = T(L_b)$ in L_c^* such that:

(1) $L_b \subseteq L_c$, $|L_b| = |L_c|$.

(2) Every model M_b for L_b has a fixed expansion to a model M_c for L_c which is a model of T_c.

(3) Every formula in L_c^* is T_c-equivalent to an atomic formula; i.e. for all $\varphi(\bar{x}) \in L_c^*$ there is a predicate symbol $R_\varphi(\bar{x})$ such that $(\forall \bar{x})(\varphi(\bar{x}) \equiv R_\varphi(\bar{x})) \in T_c$.

(4) T_c has Skolem functions; i.e., for all $\varphi(y, \bar{x}) \in L_c^*$ there is a function symbol $F_\varphi \in L_c^*$ such that

$$(\forall \bar{x})[(\exists y)\varphi(y, \bar{x}) \equiv \varphi(F_\varphi(\bar{x}), \bar{x})] \in T_c.$$

(5) For every formula $\varphi(x, y, \bar{z}) \in L_c^*$ there are function symbols $F_\varphi^i \in L_c$ (for $i = 1, \cdots, 5$) such that: if $|M_b| = \lambda_0$ (the universe set of M_b), $<^{M_b}$ is the "natural" order, then for all sequences $\bar{a}$ from M_b if $\varphi(x, y, \bar{a})$ linearly orders $A = \{y \in |M_c|: M_c \models (\exists x)\varphi(x, y, \bar{a})\} \neq \emptyset$ then (in M_c):

(i) $F_\varphi^1(\bar{a}) = \mathrm{cf}(A, \varphi(x, y, \bar{a}))$.

(ii) The sequence $\langle F_\varphi^2(y, \bar{a}): y < F_\varphi^1(\bar{a})\rangle$ is an increasing unbounded sequence in A.

(iii) A has a cut (A_1, A_2) such that $\mathrm{cf}^*(A_2, \varphi(x, y, \bar{a})) = \mu$, $\mathrm{cf}(A_1, \varphi(x, y, \bar{a})) = \chi$ iff $F_\varphi^3(\mu, \chi, \bar{a}) = 0$ iff $F_\varphi^3(\mu, \chi, \bar{a}) \neq 1$.

(iv) If $F_\varphi^3(\mu, \chi, \bar{a}) = 0$ then $\langle F_\varphi^4(y, \mu, \chi, \bar{a}): y < \chi\rangle$ is an increasing unbounded sequence in A_1.

(v) If $F_\varphi^3(\mu, \chi, \bar{a}) = 0$ then $\langle F_\varphi^5(y, \mu, \chi, \bar{a}): y < \mu\rangle$ is a decreasing unbounded sequence in A_2 [where A_1, A_2 in (iv), (v) are from (iii)].

PROOF. If in each stage we were to take $\varphi \in L_b^*$ (instead of L_c^*) the proof would be trivial. By repeating this process ω times we get the desired result.

Notation. Define languages L_n and theories T_n in L_n^* as follows: $L_0 = L_a \cup \{P\}$ where L_a is from Assumption 3 and P is a new unary predicate symbol. If L_n is defined let $L_n' = L_n \cup \{P_n, P^n\}$ where P_n, P^n are new unary predicate symbols. Now L_{n+1}, T_{n+1} will be L_c and $T(L_b)$ from Claim 1.4 where L_n' corresponds to L_b. Clearly L_n are countable. Let $L_\infty = \bigcup L_n$, $T_\infty = \bigcup T_n$.

DEFINITION 1.4. If M is a model, Δ a set of formulas $\varphi(\bar{x})$ (i.e. a formula with a finite sequence of variables, including its free variables) in the language of M, $A \subseteq |M|$, then the sequence $\{b_i: i < \alpha\} \subseteq |M|$ is *Δ-indiscernible* (or a sequence of *Δ-indiscernibles*) *over* A if $i \neq j \Rightarrow b_i \neq b_j$ and for all $\varphi(x_0, \cdots, x_{k-1}) \in \Delta$, $n \leqslant k$, permutation σ of $\{0, \cdots, n-1\}$ and

$a_n, \cdots, a_{k-1} \in A$ and $j(0) < \cdots < j(n-1) < \alpha$, $i(0) < \cdots < i(n-1) < \alpha$ the following holds:

$$M \models [b_{i(\sigma(0))}, \cdots, b_{i(\sigma(n-1))}, a_n, \cdots, a_{k-1}]$$
$$\Longleftrightarrow M \models [b_{j(\sigma(0))}, \cdots, b_{j(\sigma(n-1))}, a_n, \cdots, a_{k-1}].$$

Claim 1.5. 1. If A, Δ, M are as in Definition 1.4, A and Δ are finite, and $B \subseteq |M|$ is infinite, then there are $b_i \in B$ such that $\{b_i: i < \omega\}$ is Δ-indiscernible over A.

2. If A, Δ, M are as in Definition 1.4, Δ is finite, $B \subseteq |M|$, $|A| < \kappa \leq |B|$, then there are $b_i \in B$ such that $\{b_i: i < \kappa\}$ is Δ-indiscernible over A (κ is the weakly compact cardinal chosen at the beginning).

PROOF. 1. This is a result of the infinite Ramsey theorem. Ehrenfeucht-Mostowski [EM] used this to obtain essentially (1).

(2) It is known that κ is weakly compact iff $\kappa \longrightarrow (\kappa)^m_\mu$ for all $\mu < \kappa$ (see [Si 1]). From here the result is immediate. □

Let $\{c_\alpha: \alpha < \alpha_T\}$ be all the individual constants in $L - L_a$ (see Assumption 3). Let $S = \{(t, n, B): t \subseteq T, n < \omega, B \subseteq \{c_\alpha: \alpha < \alpha_T\}, t$ and B finite$\}$. Denote elements of S by s or $s_i = (t_i, n_i, B_i)$ and $s_1 \leq s_2$ will mean $t_1 \subseteq t_2$, $n_1 \leq n_2$, $B_1 \subseteq B_2$. Now we define the L_n-model $M(s)$, $s = (t, n, B)$. For t, B fixed, denote $M(s)$ by M^n. Define M^n by induction on n such that M^{n+1} expands M^n, M^n is an L_n-model, $P_n(M^{n+1}) \subseteq \omega$, $P^n(M^{n+1}) \subseteq \kappa$, $|P_n(M^{n+1})| = \aleph_0$, $|P^n(M^{n+1})| = \kappa$. For $n = 0$ take M^0 to be the expansion of M_t by adding the predicate $P(M^0) = B$. Let $\{\varphi_i(\bar{x}^i): i < \omega\}$ be a list of the formulas of L_∞, such that the number of variables in $\bar{x}^i$ is $\leq i$, and let $\Delta_n = \{\varphi_i: i \leq n\} \cap L_n$. If M^n is defined we define M^{n+1} as follows: Let $A^1 \subseteq P^{n-1}(M^n)$ (or $A^1 \subseteq \{a: a < \kappa\}$ if $n = 0$) be a Δ_n-indiscernible sequence over $B \cup \{a: a < \omega\}$ and let $A^2 \subseteq P_{n-1}(M^n)$ (or $A^2 \subseteq \{a: a < \omega\}$ if $n = 0$) be a Δ_n-indiscernible sequence over $B \cup \{a^1, \cdots, a^n\}$, where $a^1, \cdots, a^n$ are the first n elements of A^1. (In fact A^1, A^2 are sets, but we look on them as sequences by the ordering $<$.) As for each $\varphi(\bar{x}) \in \Delta_n$ the number of variables in $\bar{x}$ is $\leq n$, A^2 is Δ_n-indiscernible over $B \cup A^1$. Expand M^n by interpreting P^n as A^1 and P_n as A^2, and then expand the result to an L_{n+1}-model by Claim 1.4, so it will be a model of T_n (mentioned in the notation after Claim 1.4). This will be M^{n+1}. Let L_U be the language obtained from L_∞ by adding the individual constants $\{c_\alpha: \alpha < \alpha_T\}$ (from $L - L_a$) and new constants y^i, y_i for $i < \kappa$. Now we define a first-order theory T_U in L_U. Let $\psi(x_1, \cdots, x_l; x^1, \cdots, x^m; z_1, \cdots, z_k)$ be a formula in L_∞ and let $j(1) < \cdots < j(m) < \kappa$, $i(1) < \cdots < i(l) < \kappa$. Then

$$\psi(y_{i(1)}, \cdots, y_{i(l)}; y^{j(1)}, \cdots, y^{j(m)}; c_{\alpha(1)}, \cdots, c_{\alpha(k)}) \in T_U$$

iff there is $s_1 \in S$ such that, for all $s \geqslant s_1$, $s = (t, n, B)$, and for all $a_1 < \cdots < a_l \in P_n(M(s))$, $b_1 < \cdots < b_m \in P^n(M(s))$, it is the case that

$$M(s) \models \psi[a_1, \cdots, a_l; b_1, \cdots, b_m; c_{\alpha(1)}, \cdots, c_{\alpha(k)}].$$

Clearly T_U is consistent. Let $M \models T_U$ be κ^+-saturated (see Morley and Vaught [MV] or e.g. Chang and Keisler [CK]). Let N be the submodel of M whose universe set is the closure of P^M under the functions of M (and so in particular all the individual constants are in N). Let D be a nonprincipal ultrafilter on ω, and let $N^* = N^\omega/D$. We shall show that $N^* \models T$, and thus complete the proof of the theorem. We use the fact that N^* is $\aleph_1$-saturated (see e.g. [CK]).

Because of Claim 1.4(3) it is sufficient to show:

(I) If $R_1(x, y, \bar{z})$ is an atomic formula in L_∞ and $(\forall \bar{z})[(Q^{\text{cf}}x, y) R_1(x, y, \bar{z}) \equiv R_2(\bar{z})] \in T_\infty$, then for all $\bar{a} \in N^*$

$$N^* \models (Q^{\text{cf}}x, y)R_1(x, y, \bar{a}) \Longleftrightarrow N^* \models R_2[\bar{a}].$$

(II) If $R_1(x, y, \bar{z})$ is an atomic formula in L_∞ and $(\forall z)[(Q^{\text{dc}}x, y) R_1(x, y, \bar{z}) \equiv R_2(\bar{z})] \in T_\infty$, then for all $\bar{a} \in N^*$

$$N^* \models (Q^{\text{dc}}x, y)R_1(x, y, \bar{a}) \Longleftrightarrow N^* \models R_2[\bar{a}].$$

Proof of (I). Clearly the sets $\{a \in N^*: a < \omega(N^*)\}$, $\{a \in N^*: a < \kappa(N^*)\}$ are linearly ordered by $<$, and both have cofinality κ. So by the assumptions and Claim 1.4(5), $N^* \models R_2(\bar{a}) \Rightarrow N^* \models (Q^{\text{cf}}x, y)R_1(x, y, \bar{a})$.

Now assume $N^* \models \neg R_2[\bar{a}]$ but $N^* \models (Q^{\text{cf}}x, y)R_1(x, y, \bar{a})$. We shall produce a contradiction. Hence $R_1(x, y, \bar{a})$ linearly orders $A = \{b: N^* \models (\exists x)R_1(x, b, \bar{a})\} \neq \emptyset$, and A has no last element. Since N^* is $\aleph_1$-saturated, $\text{cf}\, A > \aleph_0$ and so by $N^* \models (Q^{\text{cf}}x, y)R_1(x, y, \bar{a})$ we have that $\text{cf}\, A = \kappa$. By the assumptions and Claim 1.4(5)(ii) we may assume that $R_1(x, y, \bar{a}) = x < y \wedge y < a$ (a is one element in place of the sequence $\bar{a}$), $N^* \models RC[a]$, and so $A = \{b: N^* \models b < a\}$. Let $\{a_i\}_{i<\kappa}$ be an increasing unbounded sequence in A, $a_\kappa = a$, and suppose that $a_i = \langle \cdots, a_i^n, \cdots \rangle_{n<\omega}/D$ where $a_i^n \in N$ (since $N^* = N^\omega/D$).

Now for all $\alpha < \beta < \kappa$ define $f(\alpha, \beta) = \{n < \omega: a_\alpha^n < a_\beta^n < a_\kappa^n, RC[a_\kappa^n], a_\kappa^n \neq \omega, \kappa\}$. Since $N^* \models (a_\alpha < a_\beta < a_\kappa \wedge RC[a_\kappa] \wedge a_\kappa \neq \omega \wedge a_\kappa \neq \kappa)$ we have by Łos' theorem that $f(\alpha, \beta) \in D$. κ, being weakly compact, satisfies $\kappa \longrightarrow (\kappa)^2_{2^{\aleph_0}}$ and so without loss of generality $f(\alpha, \beta) = f(0, 1)$. If, for all $n \in f(0, 1)$, there exists b^n such that $a_\alpha^n < b^n < a_\kappa^n$ for all $\alpha \in \kappa$, then $b = \langle \cdots, b^n, \cdots \rangle/D \in N^*$ and $a_\alpha < b < a$ for all $\alpha < \kappa$, a contradiction.

So there is $n \in f(0, 1)$ for which $\{a_\alpha^n: \alpha < \kappa\}$ is an (increasing) unbounded sequence in $\{b \in N: b < a^n\}$ and $N \models RC[a^n] \wedge a^n \neq \omega \wedge a^n \neq \kappa$. From now on denote $a = a^n, a_\alpha = a_\alpha^n$. Let $a_\alpha = \tau_\alpha(\cdots, y^{j(\alpha,m)}, \cdots; \cdots, y_{i(\alpha,l)}, \cdots; \bar{b}_\alpha)_{l<l(\alpha),m<m(\alpha)}$, where τ_α is a term, in L_∞, $j(\alpha, m)$ is an increasing sequence in m, $i(\alpha, l)$ is an increasing sequence in l, and $\bar{b}_\alpha$ is a sequence from P^N. Since we may replace $\{a_\alpha: \alpha < \kappa\}$ by any subset of the same power, we may assume that $m(\alpha) = m_0$. $l(\alpha) = l_0$, and $\tau_\alpha = \tau$ for all $\alpha < \kappa$.

Since $N \models RC[a] \wedge a > \omega$ and in every $M(s)$ the interpretation of P is a finite set, and $\{b: b < \omega\}$ is a countable set, there is a function symbol F in L_∞ such that

$$F(x^0, \cdots, x^{m_0-1}, x) = \sup\{\tau(x^0, \cdots, x^{m_0-1}; z_0, \cdots, z_{l_0-1}, v_1, \cdots) < x: z_0, \cdots, < \omega, v_1, \cdots, \in P\}.$$

Clearly $\tau(\cdots, y^{j(\alpha,m)}, \cdots; \cdots, y_{i(\alpha,l)}, \cdots; \bar{b}_\alpha) < F(\cdots, y^{j(\alpha,m)}, \cdots, a) < a$, and thus without loss of generality $a_\alpha = F(\cdots, y^{j(\alpha,m)}, \cdots, a)$. If $N \models a < \kappa$ then N satisfies the sentence "saying:" there is a regular cardinal $a < \kappa$ such that X_κ is an unbounded subset of $\{c: c < a\}$, but X_b is a bounded subset of $\{c: c < a\}$ for any $b < \kappa$; where $X_b = \{F(\cdots, x, \cdots, a) < a: x < b\}$. Hence, for some $s, M(s)$ satisfies it, contradicting the fact that $\operatorname{cf} \kappa = \kappa$. If $N \models a > \kappa$, as we get F we can get F' such that $a_\alpha < F'(a) < a$ for every α, a contradiction.

PROOF OF (II). As in the proof of (I) it is clear by Claim 1.4 that $N^* \models R_2[\bar{a}] \Rightarrow N^* \models (Q^{\mathrm{dc}}x, y)R_1(x, y, \bar{a})$.

Now assuming $N^* \models (Q^{\mathrm{dc}}x, y)R_1(x, y, \bar{a}) \wedge \neg R_2(\bar{a})$ we shall arrive at a contradiction. We can restrict ourselves to the case where $x < y \equiv_{\mathrm{def}} R_1(x, y; \bar{a})$ linearly orders $A = \{b \in N^*: (\exists x)R_1(x, b, \bar{a})\} \neq \varnothing$, A has no last element. Since there are pairing functions we may replace $\bar{a}$ by a. By hypothesis A has a Dedekind cut (A_1, A_2) such that $\operatorname{cf} A_1, \operatorname{cf}^* A_2 \in \{\omega, \kappa\}$.

Case 1. $\operatorname{cf} A_1 = \operatorname{cf}^* A_2 = \omega$: This contradicts the $\aleph_1$-saturation of N^*.

Case 2. $\operatorname{cf} A_1 = \omega, \operatorname{cf}^* A_2 = \kappa$: Let $\{b_m\}_{m<\omega}$ be an increasing unbounded sequence in A_1, and let $\{a_\alpha\}_{\alpha<\kappa}$ be a decreasing unbounded sequence in A_2, where $b_m = \langle \cdots, b_m^n, \cdots \rangle_{n<\omega}/D, a_\alpha = \langle \cdots, a_\alpha^n, \cdots \rangle_{n<\omega}/D$.

For all $\alpha < \kappa$ define $f_1(\alpha) = \langle \{n < \omega: b_m^n < a_\alpha^n\}: m < \omega \rangle$. Since the range of f_1 is a set of power $\leqslant 2^{\aleph_0}$ we can assume that f_1 is constant. Let $T_m = \{n < \omega: b_m^n < a_\alpha^n\}$; clearly $T_m \in D$. Let R be a new one-place predicate symbol, $R^n = \{b_m^n: n \in T_m\}$, and $(N^*, R) = \Pi_{n<\omega}(N, R^n)/D$. Clearly $\{b_m: m < \omega\} \subseteq R \cap A$ and $\langle R \cap A, <^* \rangle$ is an $\aleph_1$-saturated model of the theory of order, and so it contains an upper bound to the b_m's, and also $b <^* a_\alpha$ for all $b \in R \cap A$, $\alpha < \kappa$. This is a contradiction.

Case 3. $\mathrm{cf}\, A_1 = \kappa$, $\mathrm{cf}^* A_2 = \omega$: The proof is similar to the proof of Case 2.

Case 4. $\mathrm{cf}\, A_1 = \mathrm{cf}^* A_2 = \kappa$: Let $\{a_\alpha\}_{\alpha<\kappa}$ ($\{b_\alpha\}_{\alpha<\kappa}$) be an increasing (decreasing) unbounded sequence in A_1 (A_2), where $a_\alpha = \langle \cdots, a_\alpha^n, \cdots \rangle_{n\in\omega}/D$, $b_\alpha = \langle \cdots, b_\alpha^n, \cdots \rangle_{n\in\omega}/D$.

As in (I) we can assume that for all $\alpha < \beta < \kappa$ the following sets are not dependent on the particular α or β:

$$J_1 = \{n < \omega: a_\alpha^n < a_\beta^n\}, \quad J_2 = \{n < \omega: a_\alpha^n < b_\beta^n\}, \quad J_3 = \{n < \omega: b_\beta^n < b_\alpha^n\}.$$

Also $J_i \in D$, and $J_0 = \{n < \omega: N \models \neg R_2[a^n]\} \in D$, where $a = \langle \cdots, a^n, \cdots \rangle$. Thus as in (I), for some $n \in \bigcap J_i$, $R_1(x, y, a^n)$ linearly orders

$$A = \{y \in N: (\exists x) R_1(x, y, a^n)\} \supseteq \{a_\alpha^n, b_\alpha^n: \alpha < \kappa\}$$

and, for no $c \in A$, $a_\alpha^n < c < b_\alpha^n$. So by renaming,

(*) There is $a \in N$, $N \models \neg R_2[a]$, $A = \{b \in N: N \models (\exists x) R_1(x, b, a)\}$ is linearly ordered by $x <^* y = R_1(x, y, a)$, and A has a cut (A_1, A_2) with $\{a_\alpha\}_{\alpha<\kappa}$ ($\{b_\alpha\}_{\alpha<\kappa}$) an increasing (decreasing) unbounded sequence in A_1 (A_2). Let

$$a_\alpha = \tau_\alpha(\cdots, y^{j(\alpha,l)}, \cdots; \cdots, y_{i(\alpha,m)}, \cdots; \bar{d}_\alpha)_{l<l(\alpha), m<m(\alpha)},$$

and $j(\alpha, l)$ and $i(\alpha, m)$ increase with l, m respectively, $a = \tau^*(\cdots, y^{\xi(l)}, \cdots; \cdots, y_{\zeta(m)}, \cdots; \bar{d})$: where $\bar{d}, \bar{d}_\alpha$ are sequences from $P^M = P^N$.

Since κ is weakly compact we can assume the following:

(1) $\tau_\alpha = \tau_0$, $l(\alpha) = l(0)$, $m(\alpha) = m(0)$.

(2) For every formula $\varphi(\bar{x}^1, \bar{x}^2, \bar{x}^3) \in L_\infty$ the truth value of $\varphi(\bar{d}_\alpha, \bar{d}_\beta, \bar{d})$ is the same for all $\alpha < \beta < \kappa$.

(3) There is $l_1 < l(0)$ such that for every $\alpha < \beta < \kappa$

$$\begin{aligned} y^{j(\alpha,0)} = y^{j(\beta,0)} < y^{j(\alpha,1)} = y^{j(\beta,1)} < \cdots < y^{j(\alpha,l_1-1)} = y^{j(\beta,l_1-1)} \\ < y^{j(\alpha,l_1)} < y^{j(\alpha,l_1+1)} < \cdots < y^{j(\alpha,l(0)-1)} < y^{j(\beta,l_1)} \\ < \cdots < y^{j(\beta,l(0)-1)} \end{aligned}$$

and $y^{\xi(l)} < y^{j(\alpha,l_1)}$ fo- any l. Denote for $l < l_1$ $y^{j(l)} = y^{j(\alpha,l)}$,

$$\bar{y}^* = \langle y^{j(0)}, \cdots, y^{j(l_1-1)}, \cdots, y^{\xi(l)}, \cdots \rangle,$$

$$\bar{y}^\alpha = \langle y^{j(\alpha,l_1)}, \cdots, y^{j(\alpha,l(0)-1)} \rangle.$$

(4) Similar to (3) for the $y_{i(\alpha,m)}$, we get $\bar{y}_\alpha$ and $\bar{y}_*$. Thus $a_\alpha = \tau_0(\bar{y}^*, \bar{y}^\alpha, \bar{y}_*, \bar{y}_\alpha, \bar{d}_\alpha)$, $a = \tau^*(\bar{y}^*, \bar{y}_*, \bar{d})$. By treating the b_α similarly and making some change in $\bar{y}^*, \bar{y}_\alpha, \bar{d}_\alpha$ we may assume

(5) $b_\alpha = \tau^0(\bar{y}^*, {}^\alpha\bar{y}, \bar{y}_*, {}_\alpha\bar{y}, \bar{d}^\alpha)$, and if $\alpha < \beta$ then every element of ${}^\alpha\bar{y}$ comes before every element of ${}^\beta\bar{y}$ (in the sequence $\{y^i: i < \kappa\}$), and after every element of $\bar{y}^*$. Similarly for ${}_\alpha\bar{y}$. (Of course $\bar{d}^\alpha$ is a sequence from P^M; $\bar{y}^*, {}^\alpha\bar{y}$ from $\{y^i: i < \kappa\}$ and $\bar{y}_*, {}_\alpha\bar{y}$ from $\{y_i: i < \kappa\}$.)

(6) As a strengthening of (2), for all $\varphi(\bar{x}^1, \bar{x}^2, \bar{x}^3) \in L_\infty$ and all α, β the truth values of $\varphi(\bar{d}^\alpha, \bar{d}^\beta, \bar{d})$, $\varphi(\bar{d}_\alpha, \bar{d}^\beta, \bar{d})$, and $\varphi(\bar{d}_\alpha, \bar{d}_\beta, \bar{d})$ are dependent only on the order between α and β.

Notation. $a_{\alpha,\beta,\gamma} = \tau_0(\bar{y}^*, \bar{y}^\alpha, \bar{y}_*, \bar{y}_\beta, \bar{d}_\gamma)$, $b_{\alpha,\beta,\gamma} = \tau^0(\bar{y}^*, {}^\alpha\bar{y}, \bar{y}_*, {}_\beta\bar{y}, \bar{d}^\gamma)$.

Notice that by the indiscernibility of the y's and (6), $a_{\alpha,\beta,\gamma}, b_{\alpha,\beta,\gamma} \in A$ and the order between $a_{\alpha,\beta,\gamma}$ and $a_{\alpha(1),\beta(1),\gamma(1)}$ depends only on the order between α and $\alpha(1)$, the order between β and $\beta(1)$, and the order between γ and $\gamma(1)$; and similarly for the $b_{\alpha,\beta,\gamma}$.

Now for every $\alpha, \beta, \gamma, \delta < \kappa$ choose ϵ, $\alpha, \beta, \gamma, \delta < \epsilon < \kappa$. So $a_\alpha < b_\epsilon \Rightarrow a_{\alpha,\alpha,\alpha} < b_\epsilon \Rightarrow a_{\alpha,\beta,\gamma} < b_\epsilon \Rightarrow a_{\alpha,\beta,\gamma} < b_\delta$, and hence every $a_{\alpha,\beta,\gamma} \in A_1$. Similarly $b_{\alpha,\beta,\gamma} \in A_2$.

If $a_{0,0,1} \leqslant a_{1,1,0}$ then $\alpha < \alpha(1), \beta > \beta(1)$ imply $a_{\alpha,\alpha,\beta} \leqslant a_{\alpha(1),\alpha(1),\beta(1)}$. So for all $\alpha > 0, a_{\alpha,\alpha,\alpha} \leqslant a_{\alpha+1,\alpha+1,0}$, and so $\{a_{\alpha,\alpha,0}: \alpha < \kappa\}$ is an unbounded subset of A_1. Similarly, if $a_{0,0,1} \leqslant a_{1,1,0}$ and $a_{1,2,0} \leqslant a_{2,1,0}$ then $\{a_{\alpha,1,0}: \alpha < \kappa\}$ is unbounded in A_1, if $a_{0,0,1} \leqslant a_{1,1,0}$ and $a_{1,2,0} > a_{2,1,0}$ then $\{a_{1,\alpha,0}: \alpha < \kappa\}$ is unbounded in A_1, and if $a_{0,0,1} > a_{1,1,0}$ then $\{a_{0,0,\alpha}: \alpha < \kappa\}$ is unbounded in A_1. A parallel claim is true for the b's. So we may change τ_0 and τ^0 such that $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$ will each be dependent only on one index. (If $a_{\alpha,\beta,\gamma}$ is not dependent on α, then $\bar{y}^\alpha$ is empty; if not dependent on β, $\bar{y}_\beta$ is empty, and if not dependent on γ, $\bar{d}_\gamma$ is constant.) There are, in all, nine possibilities.

We shall now show that there cannot be dependence on γ alone. Assume without loss of generality that $a_\alpha = \tau_0(\bar{y}; \bar{d}_\gamma)$ where $\bar{y}$ is the concatenation of all sequences from $\{\bar{y}_i, \bar{y}^i: i < \kappa\}$ which are not dependent on γ. Consider the following type in the variables x_i, $i < l = l(\bar{d}_\gamma)$: (let $\bar{x} = \langle x_1, \cdots, x_l\rangle$: $\{P(x_i): i < l\} \cup \{(\exists x)R_1(x, \tau_0(\bar{y}, \bar{x}), a)\} \cup \{\tau_0(\bar{y}, \bar{x}) < b_\alpha: \alpha < \kappa\} \cup \{a_\alpha < \tau_0(\bar{y}, \bar{x}): \alpha < \kappa\}$).

This type, containing parameters from N, is finitely satisfiable in N and thus in M since N is an elementary submodel of M. Thus it is satisfiable by $\bar{c} = \langle c_1, \cdots, c_l\rangle$ in M, since M is κ^+-saturated. But $c_i \in N$ since $c_i \in P^M$ and thus the type is satisfiable in N. This contradicts the definition of the a_α, b_α.

We are left with four cases. Without loss of generality we shall deal only with the case $a_\alpha = \tau_0(\bar{y}^*, \bar{y}^\alpha, \bar{y}_*, \bar{d}), b_\alpha = \tau^0(\bar{y}^*, \bar{y}_*, {}_\alpha\bar{y}, \bar{d})$. Without loss of generality all the above sequences are of equal length, and it will be recalled that the sequences of the y's here are increasing sequences, $\bar{y}^* < \bar{y}^\alpha, \bar{y}_* < {}_\alpha\bar{y}$ (i.e., every element in the left sequence is smaller than every element in the matching right sequence).

For every sentence ψ which N satisfies and $s_1 \in S$ there is $s \geqslant s_1$ such that $M(s)$ satisfies ψ. Hence there are $s \in S$, and a sequence $\bar{d}' \in P[M(s)]$ where $s = (t, n, B)$ such that $n > 1000l(y^*)$ and n is big enough so that all the formulas we shall need are in Δ_{n-3} and (remembering the indiscernibility in the definition of $P^{n-2}[M(s)], P_{n-2}[M(s)]$).

(**) If $\bar{c}^* < \bar{c}^1 < \bar{c}^2$ are increasing sequences from $P^{n-2}[M(s)]$ and $\bar{c}_* < {}_1\bar{c} < {}_2\bar{c}$ are increasing sequences from $P_{n-2}[M(s)]$, and $l(\bar{c}^*) = l(\bar{y}^*)$, $l(\bar{c}^2) = l(\bar{c}^1) = l(\bar{y}^1), l(\bar{c}_*) = l(\bar{y}_*), l({}_1\bar{c}) = l({}_2\bar{c}) = l({}_1\bar{y})$ then

(A) $M(s) \models \neg R_2[\tau(\bar{c}^*, \bar{c}_*, \bar{d}')], R_1(x, y, \tau(\bar{c}^*, \bar{c}_*, \bar{d}'))$ is a linear order $<^*$ (nonempty) without a last element on a set A_s.

(B) In $M(s)$ the following holds:

$$\tau_0(\bar{c}^*, \bar{c}^1, \bar{c}_*, \bar{d}') <^* \tau_0(\bar{c}^*, \bar{c}^2, \bar{c}_*, \bar{d}') <^* \tau^0(\bar{c}^*, \bar{c}_*, {}_2\bar{c}, \bar{d}') <^* \tau^0(\bar{c}^*, \bar{c}, {}_1\bar{c}, \bar{d}') \in A_s.$$

Define $A_s^1 = \{b \in A_s$: there is $\bar{c}^0 > \bar{c}^*$ such that $b <^* \tau_0(\bar{c}^*, \bar{c}^0, \bar{c}_*, \bar{d}')\}$ and $A_s^2 = \{b \in A_s$: there is $\bar{c}_0 > \bar{c}_*$ such that $\tau^0(\bar{c}^*, \bar{c}_*, \bar{c}_0, \bar{d}')\} <^* b$. Clearly $A_s^1 \cap A_s^2 = \emptyset$, cf $A_s^1 = \kappa$, cf $A_s^2 = \omega$, but from $M(s) \models \neg R_2[\tau(\bar{c}^*, \bar{c}_*, \bar{d}')]$ and by the definition of R_2 it follows that $M(s) \models \neg(Q^{\mathrm{dc}}x, y)R_1[x, y, \tau(\bar{c}^*, \bar{c}, \bar{d}')]$. Thus there is $b \in A_s, A_s^1 < b < A_s^2$. But A_s, A_s^1, A_s^2 are definable by the formulas $\varphi(x, \bar{c}^*, \bar{c}, \bar{d}'), \varphi^1(x, \bar{c}^*, \bar{c}_*, \bar{d}'), \varphi^2(x, \bar{c}^*, \bar{c}_*, \bar{d}')$, where $\varphi, \varphi^1, \varphi^2 \in L_n$.

Now by 1.4 there is a function symbol F in L_{n+1} such that for all s_1 such that $n_1 > n$ the following sentence holds in $M(s_1)$ (abusing our notation the free variables are $\bar{y}_*, \bar{y}^*, \bar{z}$):

If $\neg R_2(\tau(\bar{y}^*, \bar{y}_*, \bar{z}))$; and $R_1(x, y, \tau(\bar{y}^*, \bar{y}_*, \bar{z}))$ defines a linear order on $A = \{v: (\exists x)R_1(x, v, \bar{z})\}$; $\bar{y}_*$ $(\bar{y}^*)$ is a sequence of elements $< \omega$ $(< \kappa)$; and for all $\bar{y}^* < \bar{y}^1 < \bar{y}^2$ such that the elements of $\bar{y}^1, \bar{y}^2$ are in P^n, and for all $\bar{y}_* < \bar{y}_1 < \bar{y}_2$ such that the elements of $\bar{y}_1, \bar{y}_2$, are in P_n, it is true that

$$\tau_0(\bar{y}^*, \bar{y}^1, \bar{y}_*, \bar{z}) <^* \tau_0(\bar{y}^*, \bar{y}^2, \bar{y}_*, \bar{z}) <^* \tau^0(\bar{y}_1, \bar{y}_*, \bar{y}_2, \bar{z}) <^* \tau^0(\bar{y}^*, \bar{y}_*, \bar{y}_1, \bar{z}) \in A$$

where $x <^* y \equiv R_1(x, y, \tau(\bar{y}^*, \bar{y}_*, \bar{z}))$, *then* $F(\bar{y}^*, \bar{y}_*, \bar{z}) \in A$ and for all $\bar{y}_1, \bar{y}^1$ as above

$$\tau_0(\bar{y}^*, \bar{y}^1, \bar{y}_*, z) <^* F(\bar{y}^*, \bar{y}_*, \bar{z}) < \tau^0(\bar{y}^*, \bar{y}_*, \bar{y}_2, \bar{z}).$$

Thus M_ι and N_ι satisfy the above sentence (because of the suitable indiscernibility of P^n, P_n). Thus $F(\bar{y}^*, \bar{y}_*, \bar{d}) \in A$, $a_\alpha < F(\bar{y}^*, \bar{y}_*, \bar{d}) < b_\alpha$, a contradiction. This concludes the proof of Theorem 1.3 and of Theorem 1.1.

2. **Discussion.** *More on* L^*. Some natural problems are:

Problem 2.1. A. In Theorem 1.2, is the condition that κ be weakly compact necessary?

B. Give L^* a "nice" axiomatization.

In Theorem 1.2 we prove actually:

THEOREM 2.2. A. L^* *satisfies the completeness theorem*; *that is, for every sentence* $\psi \in L^*$ *we can find* (*recursively*) *a recursive set* Γ *of first-order sentences* (*or even a single sentence*) *in a richer language such that* ψ *has a model iff* Γ *has a model.*

B. *Every* L-*model has* L^*-*elementary extensions of arbitrary large power.*

Clearly L^* is interpretable in L_{κ^+,κ^+} (the language with conjuction on κ formulas and quantification on κ variables), and by Hanf [Ha 1] every L-model has an L_{κ^+,κ^+}-elementary submodel of power $\leqslant |L|^\kappa$. Thus

THEOREM 2.3. A. *If* $|L| \leqslant \lambda = \lambda^\kappa$, *then every* L-*model of power* $\geqslant \lambda^\kappa$ *has an* L^*-*elementary submodel of power* λ^κ. (*If* $|L| \leqslant \kappa$ *we can choose* $\lambda = 2^\kappa$.)

B. *There is a sentence in* L^* (*having a model*) *whose models are of power* $\geqslant 2^{\aleph_0}$. *There is a consistent theory in* L^* *of power* κ *whose models are of power* $\geqslant 2^\kappa$.(1)

C. *Every consistent theory in* L^* *of power* $<\kappa$ *has a model of power* $\leqslant \kappa$.

PROOF. A has already been proved.

B is proved by the sentence "$<$ is a linear order, in which every element has immediate predecessor and successor; $\neg(Q^{\mathrm{dc}}x, y)(x < y)$; P is a nonempty convex subset, bounded from above and below, which has no first or last element." Every model of this sentence is of power $\geqslant 2^{\aleph_0}$.

Let T be the following theory:

(1) "$<$ is a linear order and $\neg(Q^{\mathrm{dc}}x, y)x < y$".

(2) "$c_i < c_j$ for all $i < j \in J$", where J is a dense κ-saturated order of power κ.

Clearly T is consistent. Now let $M \models T$ and let (J_1, J_2) be a cut of J, $\mathrm{cf}\, J_1 = \mathrm{cf}^*\, J_2 = \kappa$. So there is an element $a \in M$, $a_i < a < a_j$, for all $i \in J_1$, $j \in J_2$. Thus $\|M\| \geqslant 2^\kappa$. This completes the proof of B.

(1) We can improve 2.3B, i.e. there is $\varphi \in L$ which has models only in cardinalities $\geqslant \kappa$; see 2.24.

C is proved like 1.3, but we do not need the P.

Elimination of the assumption of the existence of a weakly compact cardinal. In place of a weakly compact cardinal we can assume:

(*) There is a proper class of regular cardinals, C_1, such that for all $\lambda \in C_1$ there are $\{S_\alpha: \alpha < \lambda^+, \mathrm{cf}\,\alpha = \lambda\}$ such that for all $S \subseteq \lambda^+$, $\{\alpha < \lambda^+: \mathrm{cf}\,\alpha = \lambda, S \cap \alpha = S_\alpha\}$ is a stationary set of λ^+.

By Jensen and Kunen [JK, §2, Theorem 1] the class of regular cardinals satisfies (*), if $V = L$.

If (*) holds we can choose C such that $\aleph_0 \in C$ $(\lambda \in C \Rightarrow \lambda^+ \notin C)$, and $C - \{\aleph_0\}$ satisfies (*).

THEOREM 2.4. *If C and (*) are as above, then $L^* = L(Q_C^{\mathrm{cf}}, Q_C^{\mathrm{dc}})$ satisfies the compactness theorem.*

PROOF. The proof is a combination of Keisler [Ke 3, §2] and Chang [Ch 2]. We assume T satisfies the conditions of 1.4, and every finite subtheory has a model. Choose $\lambda \in C$, $\lambda \geqslant |T|$ (or even $\lambda \geqslant |T|$). By (*) clearly $\lambda^\lambda = \lambda$. Now we define an increasing elementary sequence of λ-saturated models $\{M_\alpha\}_{\alpha<\lambda^+}$, such that for $\alpha < \beta, M_\beta$ is an end extension of M_α, and $M = \bigcup M_\alpha$. Also, if $a \in RC^M$ then

$$M \models (Q_C^{\mathrm{cf}} x, y)(x < y < a) \Longleftrightarrow \lambda = \mathrm{cf}\{b \in M: b < a\}$$

$$\Longleftrightarrow \lambda^+ \neq \mathrm{cf}\{b \in M: b < a\};$$

and if (A_1, A_2) is a cut of an order in M which is definable(2) (in M by a formula with parameters) such that $\mathrm{cf}\, A_1 = \lambda^+$ or $\mathrm{cf}^* A_2 = \lambda^+$ then A_1 is also definable (in M by a formula with parameters). Clearly $M \models T$. □

Cofinality quantifiers. We shall deal with logics containing just the generalized quantifier Q^{cf}. We write Q_λ^{cf} in place of $Q_{\{\lambda\}}^{\mathrm{cf}}$.

THEOREM 2.5. *Let M be an L-model of power $> \kappa$. Then M has an L^{**}-elementary submodel of power κ where $L^{**} = L(Q_{C_i}^{\mathrm{cf}}, Q_{\lambda_j}^{\mathrm{cf}})_{i<n,j<\mu}$ if*

(1) $\lambda_j \leqslant \kappa$, $|L| + \mu \leqslant \kappa$,

(2) *for every $i < n$ there are regular cardinals $\chi_1^i < \cdots < \chi_{m(i)}^i$ such that if for every l $\chi < \chi_l^i \Longleftrightarrow \chi' < \chi_l^i$ then $\chi \in C_i \Longleftrightarrow \chi' \in C_i$; and*

(3) *for all regular λ there is a regular $\lambda' \leqslant \kappa$ such that $\lambda' \neq \lambda_j$ for all j and $\lambda \in C_i \Longleftrightarrow \lambda' \in C_i$.*

PROOF. The proof is by induction on $\lambda = \|M\|$. As in §1 we can assume that $|M|$ is an ordinal, say $\lambda + 1, <^M$ is the order on the ordinals, RC^M is the set of regular cardinals in M, M has Skolem functions, and also cofinality

(2) We assume the *order* is definable.

Skolem functions (see 1.4(5)). Thus in order that a submodel N of M be an L^{**}-elementary submodel; for all $a \in RC^N$ we must have

$$M \models (Q^{\mathrm{cf}}_{\lambda_j} x, y)(x < y < a) \Longleftrightarrow N \models (Q^{\mathrm{cf}}_{\lambda_j} x, y)(x < y < a),$$

$$M \models (Q^{\mathrm{cf}}_{C_i} x, y)(x < y < a) \Longleftrightarrow N \models (Q^{\mathrm{cf}}_{C_i} x, y)(x < y < a).$$

Case 1. λ is a regular cardinal: Choose regular $\lambda' < \lambda, \lambda' \neq \lambda_j$ for all j, and $\lambda \in C_i \Longleftrightarrow \lambda' \in C_i$. Build an increasing sequence $\{M_\alpha\}_{\alpha<\lambda'}$ of elementary submodels of M such that

(i) $M_\alpha \subsetneq M_{\alpha+1}, M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ for δ a limit ordinal, $\|M_0\| \geqslant \kappa$.

(ii) $|M_\alpha|$ is an initial segment of λ with the addition of λ (which is the last element of M). $M_{\lambda'}$ will be the desired model.

Case 2. λ is singular. Choose regular $\chi < \lambda$ such that $\lambda < \chi^j_l \Longleftrightarrow \chi < \chi^j_l$. There is such a χ since the number of χ^j_l is finite and they are regular thus $\neq \lambda$, and λ is a limit cardinal. Let M_0 be an elementary submodel of M of power $\chi' = \chi^+ + \mathrm{cf}\,\lambda$ which contains $\{\alpha: \alpha \leqslant \chi'\} \cup \{\lambda\}$. Define by induction on $\alpha \leqslant \chi^+$ an increasing sequence of elementary submodels of M, $\{M_\alpha\}_{\alpha<\chi^+}$, such that $\|M_\alpha\| = \chi', M_\delta = \bigcup_{i<\delta} M_i$ for δ a limit ordinal, and if $a \in RC^M, \chi < a$, then there is $a' < a, a' \in M_{\alpha+1}$, such that for every $b < a$ if $b \in M_\alpha$, then $b < a'$. Clearly if $a \in RC^M \cap |M_{\chi'}|$ then the cofinality of $\{b \in M_{\chi'}: b < a\}$ is either χ^+ or the cofinality of $\{b \in M: b < a\}$. Thus $M_{\chi'}$ is an L^{**}-elementary submodel of M.

We may assume now that in the definition of L^{**} the C_i are pairwise disjoint.

THEOREM 2.6. *Assume* $\mu < \aleph_0$ *in the definition of* L^{**} *in* 2.5.

(A) L^{**} *satisfies the completeness theorem and the compactness theorem* (*and thus the upward Lowenheim-Skolem theorem*).

(B) *Let* T *be a theory in* $L(Q^{\mathrm{cf}}_{C_i}, Q^{\mathrm{cf}}_{\lambda_j})$. *By substituting* λ'_j *for* λ_j *and* C'_i *for* C_i *we get a theory* T'. T *has a model iff* T' *has a model, on condition that*:

(1) $\lambda_{j_1} = \lambda_{j_2} \Longleftrightarrow \lambda'_{j_1} = \lambda'_{j_2}$,

(2) $\lambda_j \in C_i \Longleftrightarrow \lambda'_j \in C'_i$,

(3) *if* $C_i = \{\lambda_{j_l}: l < l_0\}$ *then* $C'_i = \{\lambda'_{j_l}: l < l_0\}$.

REMARK. In the completeness theorem we consider a single sentence and the set of quantifiers appearing in it, so there is no need for $\mu < \aleph_0$.

SKETCH OF PROOF. Let T be a theory in L^{**}. Without loss of generality T has Skolem functions, there is a symbol $<$ which is an order on the universe, RC is a unary predicate, there are cofinality Skolem functions (see 1.4(5)), and every formula is equivalent to an atomic formula. By adding cofinality quantifiers we can assume that $L^{**} = L(Q^{\mathrm{cf}}_{C_i})_{i \leq n}$ where the C_i are disjoint intervals of regular cardinals, $C_n = \{\lambda: \lambda_0 \leq \lambda \text{ regular}\}$; $\bigcup_i C_i$ is all the regular cardinals. By using the previous theorem and the set of sentences from Shelah [Sh 2, §4], we get: if every finite $t \subseteq T$ has a model, then $T \cap L$ has a model M for which if $(\forall \bar{x})[R^i(z) \equiv (Q^{\mathrm{cf}}_{C_i} x, y)(x < y < z)] \in T$ and $M \models R^i(z) \wedge RC[z]$, then $\mathrm{cf}\{a: a < z\} = \lambda^i$.(3) From here, by [Sh 2, §4], the theorem is immediate.

Problem 2.7. When in general is L^{**} compact?

REMARK. If there is a C_i which is an infinite set of λ_j's then L^{**} is not compact. On the other hand, by the previous theorem and ultraproducts, if every finite $t \subseteq T$ has a model, then there is a T', as in (B) of the previous theorem, which has a model.

Problem 2.8. Give a nice axiomatization of L^{**}. In one case we have

THEOREM 2.9. *If $C \neq \emptyset$, and C is not the class of all regular cardinals, then the following system of axioms is complete for $L(Q^{\mathrm{cf}}_C)$:*

(1) *The usual schemes for the first order calculus.*

(2) *The following scheme (in which variables serving as parameters are not explicitly mentioned):*

$$(Q^{\mathrm{cf}}_C x, y)\varphi(x, y) \longrightarrow [\varphi(x, y) \text{ is a linear order on } \{y: (\exists x)\varphi\} \text{ without last element}]$$

$$\begin{aligned}&(Q^{\mathrm{cf}}_C x, y)\varphi(x, y) \wedge \neg (Q^{\mathrm{cf}}_C x, y)\psi(x, y) \wedge [\psi(x, y) \text{ is a linear order on } \{y: (\exists x)\psi\} \text{ without last element}]\\ &\quad \wedge (\forall x, y)[\theta(x, y) \longrightarrow (\exists x_1)\varphi(x_1, x) \wedge (\exists y_1)\psi(y_1, y)]\\ &\quad \wedge (\forall y)[(\exists y_1)\psi(y_1, y) \longrightarrow (\exists x)\theta(x, y)] \longrightarrow\\ &\neg [(\forall x_0)(\exists y_0)((\exists x)\varphi(x_1 x_0) \longrightarrow (\exists y)\psi(y, y_0)\\ &\quad \wedge (\forall x_1, y_1)(\psi(y_0, y_1) \wedge \theta(x_1, y_1) \longrightarrow \varphi(x_0, x_1)))]\end{aligned}$$

PROOF. By the previous theorem it is sufficient to prove that if $T \subseteq L(Q^{\mathrm{cf}}_C)$ is countable, complete, and consistent (by the above axiomatization), then T has a model where we interpret C as $\{\aleph_0\}$ for example. The proof is like [KM].

A quantifier close to the quantifiers we have discussed is

DEFINITION 2.1. $(Q^{\mathrm{ec}} x, y)[\varphi(x, y), \psi(x, y)]$, which means that the orders defined by $\varphi(x, y)$ and $\psi(x, y)$ on $\{y: (\exists x)\varphi(x, y)\}$ and $\{y: (\exists x)\psi(x, y)\}$, respectively, have the same cofinality.

CONJECTURE 2.10. The logic $L(Q^{\mathrm{ec}})$ is compact and complete (and even has an axiomatization parallel to that of the last theorem). It is not hard to see that

(3) The λ^i's are arbitrary.

THEOREM 2.11. (1) *There is* $\psi \in L(Q^{\mathrm{ec}})$ *which has a model of power* $\aleph_\alpha$ *iff* $\aleph_\alpha = \alpha$.

(2) *If* $\|M\| = \kappa$ *where* κ *is a Mahlo number of rank* $\alpha + 1$, *then* M *has an* $L(Q^{\mathrm{ec}})$*-elementary submodel of power* λ *for some Mahlo number* $\lambda < \kappa$ *of rank* α (*actually the set of such* λ*'s which corresponds to* M *is a stationary set*). (*For information about Mahlo numbers see Lévy* [Le 1].)

(3) *If* κ *is not a Mahlo number then there is a model of power* κ, *with a finite number of relations, which has no* $L(Q^{\mathrm{ec}})$*-elementary submodel of smaller power.*

Generalized second-order quantifiers. Henkin [Hn 1] defined first-order generalized quantifiers as follows: The truth value of $(Qx)\varphi(x)$ in a model M is dependent only on the isomorphism type of $(|M|, \{x: \varphi(x)\})$, i.e., on the powers of $\{x: \varphi(x)\}$ and $\{x: \neg\varphi(x)\}$. This is how the quantifier $(Q^{\mathrm{cr}}_\lambda x)\varphi(x) \Longleftrightarrow |\{x: \varphi(x)\}| \geqslant \lambda$ was reached.

Similarly we may define "generalized second-order quantifier" to be such that the truth value of $(QP)\varphi(P)$ in M is dependent only on the isomorphism type of $(|M|, \{P: \varphi(P)\})$, like [Li 1].

The regular second-order quantifier is too strong from the point of view of model theory, and so there are no nice model theoretic theorems about it. But there could be generalized second-order quantifiers which are weak enough for their model theory to be nice, for example by satisfying Lowenheim-Skolem, compactness or completeness theorems. In fact the cofinality quantifiers we discussed previously are an example.

DEFINITION 2.2. If $<$ is a linear order on A then an *initial segement* of A is a set $B \subsetneq A$ such that $b < a, a \in B \rightarrow b \in B$. An increasing sequence $\{B_\alpha: \alpha < \lambda\}$ of initial segments is *unbounded* if every initial segment of A is contained in some B_α, and it is *closed* if $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$ for all limit ordinals δ.

If $\mathrm{cf}\, A > \omega$ then the closed and unbounded sequences of initial segments of A generate a (nonprincipal) filter $D(A)$ on the set of all initial segments of A, $H(A)$.

Now we define some generalized second-order quantifiers.

DEFINITION 2.3. Let C be a class of regular cardinals $> \aleph_0$.

$$(Q^{\mathrm{st}}_C P, x, y)[\varphi(x, y), \psi(P)] \Longleftrightarrow (Q^{\mathrm{cf}}_C x, y)\varphi(x, y) \text{ and}$$

if $A = \{y: (\exists x)\varphi(x, y)\}$ then $H(A) - \{P: \psi(P), P \in H(A)\} \notin D(A)$; that is, the above set is stationary.

DEFINITION 2.4. Let $\lambda > \aleph_0$ be regular, and let $C \subseteq \lambda$.

$$(Q^{st}_{\lambda,C}, P, x, y)[\varphi(x, y), \psi(P)] \Longleftrightarrow (Q^{st}_\lambda P, x, y)[\varphi(x, y), \psi(P)] \text{ and}$$

there is a sequence $\{P_i\}_{i<\lambda}$ of initial segments of $\{y: (\exists x)\varphi(x, y)\}$ which is closed and unbounded, and $\{i < \lambda: \psi(P_i)\} \cup (\lambda - C) \in D(\lambda)$.(4)

REMARK. It is not difficult to see that the above is well defined, for if $\{P'_i\}_{i<\lambda}$ is another example of such a sequence $\{i: P_i = P'_i\} \in D(\lambda)$.

In another example we use a filter similar to that of Kueker **[Ku 1]**:

For a regular power $\lambda > \aleph_0$ and set A, $|A| \geqslant \lambda$, let $S_\lambda(A) = \{B: B \subseteq A, |B| < \lambda\}$. $D_\lambda(A)$ will be the filter on $S_\lambda(A)$ generated by the families $S \subseteq S_\lambda(A)$ satisfying

(1) for all $B \in S_\lambda(A)$ there is $B' \in S$ such that $B \subseteq B'$, and

(2) S is closed under increasing unions of length $< \lambda$.

Thus for example if M is a model $\|M\| > \lambda$ whose language is of power $< \lambda$ then $\{|N|: N \prec M, \|N\| < \lambda\} \in D_\lambda(|M|)$.

We can define a suitable quantifier:

DEFINITION 2.5. $(Q^{ss}_\lambda P, x)[\varphi(x), \psi(P)] \Longleftrightarrow S_\lambda(A) - \{P: |P| < \lambda, P \subseteq A \models \psi[P]\} \notin D_\lambda(A)$ where $A = \{x: \varphi(x)\}$.

Again it is not hard to check that the definition is valid.

Problem 2.12. Investigate the logics with the quantifiers (A) $Q^{st}_{\lambda,A}$; (B) Q^{st}_C; (C) Q^{ss}_λ. In particular in regard to (1) compactness theorems; (2) downward Lowenheim-Skolem theorems; (3) and transfer theorems (from one λ to another). If necessary use $V = L$.

We now mention several partial results in this context.

THEOREM 2.13. (A) *If* $\|M\| = \kappa$, κ *weakly compact,* $|L(M)| < \kappa$, C *is the class of all regular cardinals* $> \aleph_0$ *then* M *has an* $L(Q^{st}_C)$*-elementary submodel of smaller power.*

(B) $(V = L.)$ *If* κ *is not weakly compact, then there is a model of power* κ, *whose language is countable, which has no proper* $L(Q^{st}_C)$*-elementary submodel.* (C *as above.*)

PROOF. (A) follows from well-known theorems in set theory.

(B) We shall prove it for regular κ; the result for a singular one follows from it.

By Jensen **[Je 1]** there is a set S of ordinals $< \kappa$ of cofinality ω such that $\kappa - S \notin D(\kappa)$ but for all $\alpha < \kappa$ of cofinality $> \omega$, $\alpha - \alpha \cap S \in D(\alpha)$. Let f be a two-place function such that for all α of cofinality ω $\{f(\alpha, n): n < \omega\}$ is an increasing sequence with limit α. We shall choose our model to be $M = (\kappa, S, f, <, \cdots, n, \cdots)$. Assume that N is an $L(Q^{st}_C)$-elementary submodel of M of smaller power. Let $\alpha = \sup\{\beta: \beta \in N\}$, then $\operatorname{cf} \alpha > \omega$ as

(4) Q^{st}_λ means $Q^{st}_{\{\lambda\}}$.

$$M \models (Q_C^{st}P, x, y)(x < y, (\exists z)(\forall v)[P(v) \equiv v < z \wedge S(z)]), \text{ and}$$

there is a closed and unbounded set $A = \{a_i: i < \mathrm{cf}\,\alpha\} \subseteq \alpha$ of type $\mathrm{cf}\,\alpha$ which is disjoint with S because $\alpha < \kappa$, $\mathrm{cf}\,\alpha > \omega$. For every $a_i \in A$, let $a_i' = \inf\{b \in N: b \geqslant a_i\}$ and $A' = \{a_i': a_i \in A\}$. Clearly in N $a_\delta' = \sup\{a_i': i < \delta\}$ for δ a limit ordinal. Thus A' is closed and unbounded in N. If $a_i' \in S$, $\mathrm{cf}(a_i') = \omega$ and so the $f(a_i', n) \in N$ converge to a_i'. So $a_i = a_i'$, contradiction to the disjointness of A and S. Thus we have

$$N \models \neg (Q_C^{st}P, x, y)[x < y, (\exists z)(\forall v)(P(v) \equiv v < z \wedge S(z))],$$

a contradiction.

In regard to the possibility that N be of power κ, by Keisler and Rowbottom [KR] (see [CK]) we can expand M such that M will be a Jonsson algebra, and that will be a contradiction. If we restrict ourselves to $\aleph_1$ we can get stronger results.

THEOREM 2.14. (A) *$L(Q_{\aleph_1}^{cr}, Q_{\aleph_1}^{st}, Q_{\aleph_1,A_i}^{st})_{i<n}$ is $\aleph_0$-compact and complete. The consistency of a sentence is just dependent on the Boolean algebra generated by $A_i/D(\aleph_1)$, and not on the particular A_i.*(5)

(B) *$L(Q_{\aleph_1}^{st}, Q_{\aleph_1,A_i}^{st})_{i<n}$ is $\aleph_1$-compact.*

(C) *If T is a theory in $L(Q_\lambda^{cr}, Q_\lambda^{st}, Q_{\lambda,B_i}^{st})_{i<n}$ and T' is the corresponding theory in $L(Q_{\aleph_1}^{cr}, Q_{\aleph_1}^{st}, Q_{\aleph_1,A_i}^{st})_{i<n}$, and $\{B_i\}$ a partition of λ, $\{A_i\}$ a partition of ω_1, $\aleph_1 - A_i \notin D_i(\aleph_1)$, $\lambda - B_i \notin D(\lambda)$ then T has a model $\Rightarrow T'$ has a model.*

PROOF. (A) Without loss of generality we shall deal with models of power $\aleph_1$ whose universe sets are ω_1.

It is not difficult to define a language L_1, $|L_1| \leqslant |L|$ such that every L-model M, $|M| = \omega_1$ can be expanded to an L_1-model M_1 such that

(1) M_1 has Skolem functions (dependent only on the formula and not on M), and every formula (including sentences) is equivalent to an atomic formula,

(2) $<$ is the order on the ordinals, and

(3) $P_i^{M_1} = A_i$.

Let T be a theory in the logic from (A) such that every finite $t \subseteq T$ has a model M^t. Let T_1 be the set of sentences of L_1 holding in M_1^t for t large enough. Define an increasing elementary sequence of countable L_1-models: N_0 will be any countable model of T_1, $N_\delta = \bigcup_{\alpha<\delta} N_\alpha$ for $\delta < \omega_1$ limit. If N_α is defined $N_{\alpha+1}$ will be an end extension of N_α (i.e. $N_{\alpha+1} \models a < b \in N_\alpha \longrightarrow a \in N_\alpha$) such that there is a first element a_α in $|N_{\alpha+1}| - |N_\alpha|$ and $a_\alpha \in P_i \Longleftrightarrow \alpha \in A_i$. The proof that this is possible is similar to Keisler [Ke 2],

(5) Of course, every model with language L has an elementary submodel of cardinality $\leqslant |L| + \aleph_1$ in this logic.

[Ke 3]. It is not difficult to check that $\bigcup_{\alpha<\omega_1} N_\alpha$ is the required model of T.

The proof of the completeness is similar, but T_1 must be defined more carefully.

(B) The proof is similar to that of (A); here $N_{\alpha+1}$ will be an *expansion* (as well as an extension) and instead of the demand that $N_{\alpha+1}$ be an end extension, we only need that for all $\delta \leqslant \alpha$ limit ordinal the type $\{a_i < x: i < \delta\} \cup \{x < a_\delta\}$ be omitted.(6)

(C) The proof is similar. □

The class K_λ. After the proof of the previous theorem it is natural to consider the following class of models which is somewhat parallel to the class of κ-like models.

DEFINITION 2.6. Let λ be regular. $M \in K_\lambda$ iff $<$ linearly orders $\{x: M \models (\exists y)(x < y \vee y < x)\}$ with cofinality λ, and there is a continuous increasing unbounded sequence $\{a_i\}_{i<\lambda}$ (i.e. for all $\delta < \lambda$ limit, the type $\{a_i < x < a_\delta : i < \delta\}$ is omitted by M).

From the previous theorem follows

THEOREM 2.15. *If* $|T| \leqslant \aleph_1$ *(T a first-order theory) and every finite* $t \subseteq T$ *has a model in some* $K_\lambda, \lambda > \aleph_0$ *then* T *has a model in* $K_{\aleph_1}$.

It is easily proven that

THEOREM 2.16. *If* $M \in K_\lambda, \mu < \lambda$ *regular,* $|L(M)| < \lambda$ *then* M *has an elementary submodel in* K_μ.

Somewhat less immediate is the following.

THEOREM 2.17. (A) *If for every* $n < \omega$ *every finite* $t \subseteq T$ *has a model in some* K_λ *for* $\lambda \geqslant \aleph_n$, *then* T *has a model in* K_λ *for all* λ.

(B) (*Completeness.*) *The set of sentences true in every model of* $K_{\aleph_{\omega+1}}$ *is recursively enumerable.*

PROOF. Without loss of generality assume that T has Skolem functions. For every ordinal α define

$$\Sigma_\alpha = \{\tau(y_{i_1}, \cdots, y_{i_n}) < y_{i_{(k+1)}} \rightarrow \tau(y_{i_1}, \cdots, y_{i_n}) < y_{(i_k+1)}: \\ \tau \text{ is a term of } L(T), i_1 < \cdots < i_n < \alpha\}.$$

It is clear that: $T \cup \Sigma_n$ is consistent for all $n \Longleftrightarrow T \cup \Sigma_\alpha$ is consistent for all $\alpha \Longleftrightarrow$ for all λ T has a model in K_λ; for if M is a model of $T \cup \Sigma_\lambda$

(6) We should first assume w.l.o.g. that our language L has a countable sublanguage L_1, such that $L - L_1$ consist of individual constants $\{c_i: i < \omega_1\}$, $P(c_i) \in T$; and every finite $t \subset T$ has a model M^t, $|M^t| = \omega_1$, P^{M^t} is finite, and in M_1^t, every limit ordinal is the universe of a submodel of M_1^t, and (1)–(3) from the proof of (A) holds.

which is the closure of $\{y_i: i < \lambda\}$ under Skolem functions, then $M \in K_\lambda$.

Thus it is sufficient to prove:

(*) For all n and all finite $\Sigma'_n \subseteq \Sigma_n$ and all $M \in K_{\aleph_n}$ there are $y_0, \cdots, y_{n-1} \in M$ satisfying Σ'_n.

We shall show by downward induction on $m < n$ that:

(**) There are

(1) $y_{m+1} < \cdots < y_{n-1}$ (when $m = n - 1$ this is an empty sequence).

(2) $a_j^m < a_i^m < y_{m+1}$ for all $j < i \leqslant \aleph_{n-m}$, $a^m_{\aleph_{n-m}} = y_{m+1}$ (except when $n = m$).

(3) For all $\delta \leqslant \aleph_{n-m}$ limit ordinal there is no x such that $a_i^m < x < a_\delta^m$ for all $i < \delta$.

(4) If τ occurs in Σ'_n, $b_1, \cdots, b_k \in \{a_i^m: i < \alpha < \aleph_{n-m}\} \cup \{y_{m+1}, \cdots, y_{n-1}\}$, then $M \models \tau(b_1, \cdots, b_k) < y_{m+1} \longrightarrow \tau(b_1, \cdots, b_k) < a^m_{\alpha+1}$ (if $m = n - 1$ we have instead $M \models \tau(b_1, \cdots, b_k) < a^m_{\alpha+1}$).

(5) $y_{m+1}, \cdots, y_n$ satisfy the corresponding formulas of Σ'_n. Now for $m = n$ choose an increasing unbounded continuous sequence $\{a_i^n\}_{i<\aleph_n}$.

Assume that we have already completed stage $m + 1$, and we shall define for m (for simplicity let $m < n$) there is a closed unbounded set $S \subseteq \{\alpha: \alpha < \aleph_{m+1}\}$ such that for $\alpha \in S$, $\sigma_1, \cdots, \sigma_l \in \{a_i^{m+1}: i < \alpha\} \cup \{y_i: m < i < n\}$, and τ which occurs in Σ'_n we have $\tau(\sigma_1, \cdots, \sigma_l) < a^{m+1}_{\aleph_{n-m}} \longrightarrow \tau(\sigma_1, \cdots, \sigma_l) < a_\alpha^{m+1}$. Choose $\alpha_0 \in S$ such that $\mathrm{cf}(S \cap \alpha) = \aleph_m$ and define $y_m = a_\alpha^{m+1}$. Let $\{\alpha_i: i < \aleph_m\}$ be an increasing unbounded continuous sequence in $S \cap \alpha$ (it is easy to verify that there is such a sequence), and let $a_i^m = a^{m+1}_{\alpha_i}$. (If $m = 0$ there is no need to choose a_i^m, and thus it was sufficient to assume that $M \in K_{\aleph_{n-1}}$.)

THEOREM 2.18. *For all $n < \omega$ there is a sentence ψ_n having a model in $K_{\aleph_n}$ but no model in $K_{\aleph_{n+1}}$.*

PROOF. ψ_n will more or less characterize $(\omega_n, <)$.

ψ_0 will say that there is a first element, every element has a successor, and every element (except the first) has a predecessor.

ψ_{n+1} will say that $\{a: a < c_i\}$ satisfies ψ_i for $i \leqslant n$ (c_i being an individual constant), $P_0, \cdots, P_n$ is a partition of the limit elements, and if $a \in P_i$ then $\langle F_i(a, x): x < c_i\rangle$ is an increasing, continuous, unbounded sequence in $\{y: y < a\}$.

Similar theorems may be proved with omitting types as in [Mo 1]. For example if T is countable and has a model in $K_{\aleph_{\omega_1}}$ omitting a type p, then for all λ T has a model in K_λ omitting p.

Problem 2.19. Prove the compactness of $K_{\aleph_n}$, for $1 < n < \omega$.

REMARK. If we relax the condition of continuity at δ of cofinality ω then we can prove this as in [Sh 1]. Since then the class is closed under ultraproducts of $\aleph_0$ models. In general it suffices to prove the $\aleph_0$-compactness of $K_{\aleph_n}$.

General questions. A general problem (which is of course not new) about abstract logic is

Problem 2.20. Find the logical connections between the following properties of the abstract logic L^*:

(A) L^* is first-order logic.

(B) ${}_\lambda L^*$ satisfies the compactness theorem for theories of power $\leq \lambda$.

(C) $= (B)_\infty L^*$ satisfies the compactness theorem.

(D) L^* satisfies the λ-downward Lowenheim-Skolem theorem. (If $\psi \in L^*$ has a model then ψ has a model of power $\leq \lambda$.)

(E) L^* satisfies the λ-upward Lowenheim-Skolem theorem. (If ψ has a model of power $\geq \lambda$, then ψ has a model of arbitrarily large power.)

(F) L^* satisfies Craig's theorem.

(G) L^* satisfies Beth's theorem.

(H) L^* satisfies the Feferman-Vaught theorems for

(1) Sum of models.

(2) Product of models.

(3) Generalized product of models.

(I) L^* satisfies the completeness theorem (assuming that the set of sentences is recursive in the language).

It is known that (A) implies the others; for $\mu < \lambda$ (C) $\rightarrow$ $(B)_\lambda \rightarrow (B)_\mu$, $(E)_\mu \rightarrow (E_\lambda)$, $(D)_\mu \rightarrow (D)_\lambda$; (F) $\rightarrow$ (G), (C) $\rightarrow$ $(E)_{\aleph_0}$, (H)(3) $\rightarrow$ (H)(2) $\rightarrow$ (H)(1). Lindenström [Li 1], [Li 2] proved (and Friedman [F 1] reproved).

$(B)_{\aleph_0} \wedge (D)_{\aleph_0} \rightarrow (A)$, $(E)_{\aleph_0} \wedge (D)_{\aleph_0} \rightarrow (A)$, $(F) \wedge (D)_{\aleph_0} \rightarrow (A)$. The method of proof is by encoding Ehrenfeucht-Fraisse games.

Special questions which look interesting to me are

Problem 2.21. Is there a logic L^* stronger than first-order logic which is $\aleph_0$-compact and satisfies Craig's theorem? Do sums of models preserve elementary equivalence for L^*?

Is there an expansion of $L(Q^{\mathrm{cr}}_{\aleph_1})$ satisfying this? Keisler and Silver showed that $L(Q^{\mathrm{cr}}_\lambda)$ does not satisfy Craig's theorem. Friedman [Fr 2] showed that Beth's theorem is also not satisfied. Similarly it is not hard to show that all the logics with the quantifiers Q^{cf}, Q^{dc}, Q^{cc}, Q^{st} (all or some of them) do not satisfy Craig's theorem, but satisfy (H)(1). Q^{ss} does not satisfy (H)(1).

Problem 2.22. Does $L(Q^{\mathrm{ss}}_{\aleph_1})$ satisfy Craig's theorem, if we restrict ourselves to models of power $\leq \aleph_1$?

Problem 2.23. Find a natural characterization for $L(Q^{\mathrm{cr}}_{\aleph_1})$. (For $L_{\infty,\omega}$, $L_{\omega_1,\omega}$, etc. Barwise [Ba 1] found one.)

LEMMA 2.24. *Let Q^1 be the quantifier $Q^{\mathrm{dc}}_{\aleph_0}$; there is a sentence ψ in $L(Q^1)$, which has only well-ordered models, and has a model of order type α for every $\alpha \geqslant 2^{\aleph_0}$. (Thus $L(Q^1)$ is not compact.)*

PROOF. Let ψ_1 say:

1. P_1, P_2, P_3, P_4 (one place predicates) are a partition of the universe.

2. $\leqslant$ is a total order of the universe, S is the successor function in P_1 and P_2 (so P_1 and P_2 are closed under S) and each P_i is a convex subset.

3. F is a one place function mapping P_3 into P_2.

4. G is a two-place function from P_3 to P_1 and

$$(\forall x \in P_3)(\forall y \in P_3)(\forall z \in P_1)[S(z) \leqslant G(x, y) \wedge x < y \equiv (\forall v \in P_2)(\exists x', y' \in P_3)$$
$$(x < x' < y' < y \wedge \varphi(x', y', v) \wedge z \leqslant G(x', y'))]$$

where $\varphi(x, y, z) = P_3(x) \wedge P_3(y) \wedge P_2(z) \wedge x < y \wedge (\forall v)(x < v < y \rightarrow z < F(v))$.

5. $(\forall z \in P_1)(\exists x, y \in P_3)(x < y \wedge G(x, y) = z)$.

6. The cofinality of P_2 is $\aleph_0$ (just say F is an anti-isomorphism from $(P_2, \leqslant)$ onto $(P_4, \leqslant)$, and

$$(Q^1xy)[(P_2(x) \vee P_4(x)) \wedge (P_2(y) \vee P_4(y)) \wedge x < y]$$
$$\wedge (Q^1xy)(P_2(x) \wedge P_2(y) \wedge x < y)$$

7. $(Q^1xy)(P_3(x) \wedge P_3(y) \wedge x < y)$.

Suppose $M \models \psi_1$ and c_n is a strictly decreasing sequence in P_1^M; let d_n $(n < \omega)$ be an increasing unbounded sequence in P_2^M, and define inductively $x_n, y_n \in P_3^M$, $x_n < x_{n+1} < y_{n+1} < y_n$, and $G(x_n, y_n) \geqslant c_n$, and $\varphi(x_{n+1}, y_{n+1}, d_n)$. For $n = 0$ use 5, for $n + 1$ use 4. So by φ's definition for no z, $x_n < z < y_n$ for every n (as then $F(z)$ cannot be defined); contradicting 7). So in every model of ψ_1, P_1 is well-ordered. Now we define by induction on α orders I_α and functions $F_\alpha : I_\alpha \rightarrow \omega$ as follows:

I_0 is $\aleph_1$-saturated order of cardinality $2^{\aleph_0}$; F_0 is constantly zero.

$I_{\alpha+1} = \{\langle i, a\rangle : i \in \omega + 1, a \in I_\alpha\}$ ordered lexicographically.

$F_{\alpha+1}(\langle i, a\rangle) = F(a) + i$ for $i < \omega$, and zero otherwise.

$I_\delta = \{\langle \alpha, a\rangle : \alpha \leqslant \delta + 1, a \in I_\alpha\}$ ordered lexicographically.

$F_{\delta+1}(\langle \alpha, a\rangle) = F_\alpha(a)$ for $\alpha < \delta$, and zero otherwise.

Now we can easily define $M^\alpha \models \psi_1$, $P_1^{M^\alpha} = 1 + \alpha$, $P_2^{M^\alpha} = \omega$, $P_3^{M^\alpha} = I_\alpha$, $F^{M^\alpha} \supset F_\alpha$, $P_4^{M^\alpha} = \omega^*$. The change to ψ is now only technical.

Added in proof. 1. Schmerl, in a preprint "On κ-like structures which embed stationary and closed unbounded subsets" proved interesting results on problems closely related to $(Q^{\mathrm{st}}_\lambda x)$.

2. The author proved that a variant of Feferman-Vaught theorem and Beth theorem implies Craig theorem. This and other results will appear.

3. Why do we use $Q^{\mathrm{cf}}_{\{\aleph_0,\kappa\}}$, $Q^{\mathrm{dc}}_{\{\aleph_0,\kappa\}}$, and not just $Q^{\mathrm{cf}}_{\{\aleph_0\}}$, $Q^{\mathrm{dc}}_{\{\aleph_0\}}$ in Definition 1.4? (Note that $Q^{\mathrm{cf}}_{\aleph_0}$ is added just for convenience.)

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL